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Lifshitz Integral in Closed Forms

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Closed forms of the m-fold Lifshitz integral are given in terms of the hypergeometric function ${}_2F_1$ and in its arguments. The applicability of each form is discussed.

The m-fold Lifshitz integral in the form

$$I_n(r, g, \alpha) = \int d^m q \int d^{d-m} p\{r + p^2 + gq^2 + \alpha q^4\}^{-n} \tag{1}$$

has been used extensively for the last two decades to study the properties of the Lifshitz point [see the references in Hornreich (1980) and Aharony et al. (1987); also see Mukamel and Luban (1978)]. In equation (1), r, g, and α are arbitrary parameters. The integral is usually evaluated in the dimensional regularization scheme using the Feynman identities, with the help of polar coordinates (Mukamel and Luban, 1978). Such techniques are tricky, and require special skills to reach the desired results. In this paper we propose another procedure to solve equation (1) in a closed and analytic form, which is also valid in both limits, i.e., $r \to 0$ and $g \to 0$, using the properties of the hypergeometric function ${}_2F_1$.

We start with the integral

$$I_1(r, g, \alpha) = \int d^m q \int d^{d-m} p\{r + p^2 + gq^2 + \alpha q^4\}^{-1} \tag{2}$$

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After performing the p-integration using formula (3.241.4) of Gradshteyn and Ryzhik (1980, p. 292),[3] we obtain

$$I_1(r, g, \alpha) = \frac{1}{2} S_{d,d-m}\Gamma(1 - \beta)\Gamma(\beta)I_c(r, g, \alpha) \tag{3}$$

where $\Gamma(\delta)$ is the usual gamma function, $\beta = 1 - (d - m)/2$, $S_{d,d-m}$ is the geometrical factor of the integration, and

$$I_c(r, g, \alpha) = \int d^m q \, \{r + gq^2 + \alpha q^4\}^{-\beta} \tag{4}$$

By means of the substitution $q^2 = x$ in (4), completing the square, and using the formula (GR p. 286, 3.197.1), we get

$$\begin{aligned} I_c(r, g, \alpha) &= \frac{1}{2} \alpha^{-\beta} \int d^{m/2}x \, (x + a)^{-\beta}(x + b)^{-\beta} \\ &= \frac{1}{2} \alpha^{-\beta} b^{-\beta} a^{m/2-\beta} B\left(\frac{m}{2}, 2\beta - \frac{m}{2}\right) {}_2F_1\left(\beta, \frac{m}{2}, 2\beta, 1 - \frac{a}{b}\right) \end{aligned} \tag{5}$$

with $a = \bar{g}/2 - \gamma$, $b = \bar{g}/2 + \gamma$, $\bar{g} = g/\alpha$, $\bar{r} = r/\alpha$, $\gamma^2 = (\bar{g}/4)^2 - \bar{r}$, and $B(\alpha, \beta)$ is the usual beta function. In our procedure, equation (5) is considered the first closed and analytic form. But due to the complicated form of the prefactor, equation (6) will not be suitable for practical use. Therefore, in the following steps the variables a and b will be eliminated one by one using the well-known properties of the hypergeometric function.

Elimination of a in equation (5) can be done by using formula (15.3.3) of Abramowitz and Stegun (1972, p.559),[4] and (5) then reads

$$I_c(r, g, \alpha) = \frac{1}{2} \alpha^{-\beta} b^{-\beta+m/2} B\left(\frac{m}{2}, 2\beta - \frac{m}{2}\right) {}_2F_1\left(2\beta - \frac{m}{2}, \beta, 2\beta, 1 - \frac{a}{b}\right) \tag{6}$$

Then the variable b in equation (6) can be eliminated using formula (AS p. 560, 15.3.16), to obtain the form

$$\begin{aligned} I_c(r, g, \alpha) = \frac{1}{2} \alpha^{\beta-m/2}\left(\frac{g}{2}\right)^{-\beta+m/2} & B\left(\frac{m}{2}, 2\beta - \frac{m}{2}\right) \\ \times {}_2F_1\left(\beta - \frac{m}{4}, \beta - \frac{m}{4} + \frac{1}{2}, \beta + \frac{1}{2}, 1 - \frac{4\bar{r}}{\bar{g}^2}\right) \end{aligned} \tag{7}$$

[3] Henceforth, formulas quoted from this reference will be identified by GR, followed by the relevant page and equation numbers.

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Equation (7) is another closed and analytic form, which could be useful to apply for actual calculations even at $r = 0$, but not with $g = 0$. Moreover, the explicit appearance of the r dependence in the last argument of ${}_2F_1$ allows us to use such an equation to calculate the higher order of integrations I_n by means of simple differentiation, for example, $I_2 = -(\partial I_1/\partial r)$.

The inapplicability of equation (7) at $g = 0$, can be resolved by means of formula (AS p. 561, 15.3.25), to give the final form

$$I_c(r, g, \alpha) = \frac{1}{2}\,\alpha^{-m/4}\left(\frac{\tilde{g} + 2\sqrt{r}}{4}\right)^{-2(\beta - m/4)} B\left(\frac{m}{2}, 2\beta - \frac{m}{2}\right)$$
$$\times\, {}_2F_1\left(2\left(\beta - \frac{m}{4}\right), \beta - \frac{m}{4} + \frac{1}{2}, \beta + \frac{1}{2}, \frac{\tilde{g} - 2\sqrt{r}}{\tilde{g} + 2\sqrt{r}}\right) \quad (8)$$

where $\tilde{g} = g/\sqrt{\alpha}$. Equation (8) is the main target in our procedure, because it gives the correct expression in the limits of $r \to 0$ and $g \to 0$, i.e.,

$$I_1(r = 0, g, \alpha)$$
$$= \frac{1}{4} S_{d,d-m}\alpha^{\beta - m/2}(g)^{-2(\beta - m/4)}\Gamma(1 - \beta)\Gamma\left(\frac{m}{2} - \beta\right)\Gamma\left(2\beta - \frac{m}{2}\right) \quad (9)$$

$$I_1(r, g = 0, \alpha)$$
$$= \frac{1}{8} S_{d,d-m}\alpha^{-m/4}(r)^{-\beta + m/4}\Gamma(1 - \beta)\Gamma\left(\frac{m}{4}\right)\Gamma\left(\beta - \frac{m}{4}\right) \quad (10)$$

Consequently, equation (8) could be very useful for studying crossover behavior and in calculating many different physical constants, such as effective exponents for susceptibility and the specific heat, etc., near the Lifshitz point.

In summary, the procedure developed here for the evaluation of the Lifshitz integral is easy to apply and can have a wide application especially in calculating physical constants in critical phenomena. Also, in addition to the forms mentioned above, several others could be derived using well-known properties of the hypergeometric function.

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